Mathematical Excalibur

Volume 16, Number 4 *February 2012*

Olympiad Corner

Below are the problems of the 2011-2012 British Math Olympiad Round 1 held on 2 December 2011.

Problem 1. Find all (positive or negative) integers n for which $n^2+20n+11$ is a perfect square.

Problem 2. Consider the numbers 1, 2, ⋯, n. Find, in terms of n, the largest integer t such that these numbers can be arranged in a row so that all consecutive terms differ by at least t.

Problem 3. Consider a circle S. The point P lies outside S and a line is drawn through P, cutting S at distinct points X and Y. Circles S_1 and S_2 are drawn through P which are tangent to S at X and Y respectively. Prove that the difference of the radii of S_1 and S_2 is independent of the positions of P, X and Y.

Problem 4. Initially there are m balls in one bag, and n in the other, where m, $n > 0$. Two different operations are allowed:

a) Remove an equal number of balls from each bag;
b) Double the number of balls in one bag.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 28, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Zsigmondy's Theorem

Andy Loo (St. Paul's Co-educational College)

In recent years, a couple of "hard" number theoretic problems in the IMO turn out to be solvable by simple applications of deep theorems. For instances, IMO 2003 Problem 6 and IMO 2008 Problem 3 are straight forward corollaries of the Chebotarev density theorem and a theorem of Deshouillers and Iwaniec respectively. In this article we look at yet another mighty theorem, which was discovered by the Austro-Hungarian mathematician Karl Zsigmondy in 1882 and which can be used to tackle many Olympiad problems at ease.

<u>Zsigmondy's theorem</u>

First part: If a, b and n are positive integers with $a>b$, gcd(a, b)=1 and $n\geq 2$, then a^n-b^n has at least one prime factor that does not divide a^k-b^k for all positive integers $k<n$, with the exceptions of:

i) 2^6-1^6 and

ii) $n=2$ and $a+b$ is a power of 2.

Second part: If a, b and n are positive integers with $a>b$ and $n\geq 2$, then a^n+b^n has at least one prime factor that does not divide a^k+b^k for all positive integers $k<n$, with the exception of 2^3+1^3.

The proof of this theorem is omitted due to limited space. Interested readers may refer to [2].

To see its power, let us look at how short solutions can be obtained using Zsigmondy's theorem to problems of various types.

<u>Example 1 (Japanese MO 2011).</u> Find all quintuples of positive integers (a,n,p,q,r) such that

$$a^n-1 = (a^p-1)(a^q-1)(a^r-1).$$

<u>*Solution.*</u> If $a\geq 3$ and $n\geq 3$, then by Zsigmondy's theorem, a^n-1 has a prime factor that does not divide a^p-1, a^q-1 and a^r-1 (plainly $n>p,q,r$), so there is no solution. The remaining cases ($a<3$ or $n<3$) are easy exercises for the readers.

<u>Example 2 (IMO Shortlist 2000).</u> Find all triplets of positive integers (a,m,n) such that $a^m+1|(a+1)^n$.

<u>*Solution.*</u> Note that $(a,m,n)=(2,3,n)$ with $n\geq 2$ are solutions. For $a>1$, $m\geq 2$ and $(a,m)\neq(2,3)$, by Zsigmondy's theorem, a^m+1 has a prime factor that does not divide $a+1$, and hence does not divide $(a+1)^n$, so there is no solution. The cases ($a=1$ or $m=1$) lead to easy solutions.

<u>Example 3 (Math Olympiad Summer Program 2001)</u> Find all quadruples of positive integers (x,r,p,n) such that p is a prime, $n,r>1$ and $x^r-1=p^n$.

<u>*Solution.*</u> If x^r-1 has a prime factor that does not divide $x-1$, then since x^r-1 is divisible by $x-1$, we deduce that x^r-1 has at least two distinct prime factors, a contradiction *unless* (by Zsigmondy's theorem) we have the exceptional cases $x=2$, $r=6$ and $r=2$, $x+1$ is a power of 2. The former does not work. For the latter, obviously $p=2$ since it must be even. Let $x+1=2^y$. Then

$$2^n=x^2-1=(x+1)(x-1)=2^y(2^y-2).$$

It follows that $y=2$ (hence $x=3$) and $n=3$.

<u>Example 4 (Czech-Slovak Match 1996).</u> Find all positive integral solutions to $p^x-y^p=1$, where p is a prime.

<u>*Solution.*</u> The equation can be rewritten as $p^x=y^p+1$. Now $y=1$ leads to $(p,x)=(2,1)$ and $(y,p)=(2,3)$ leads to $x=2$. For $y>1$ and $p\neq 3$, by Zsigmondy's theorem, y^p+1 has a prime factor that does not divide $y+1$. Since y^p+1 is divisible by $y+1$, it follows that y^p+1 has at least two prime factors, a contradiction.

Remark. Alternatively, the results of Examples 3 and 4 follow from Catalan's conjecture (proven in 2002), which guarantees that the only positive integral solution to the equation $x^a-y^b=1$ with $x,y,a,b>1$ is $x=3$, $a=2$, $y=2$, $b=3$.

Example 5 (Polish MO 2010 Round 1). Let p and q be prime numbers with $q>p>2$. Prove that $2^{pq}-1$ has at least three distinct prime factors.

Solution. Note that 2^p-1 and 2^q-1 divide $2^{pq}-1$. By Zsigmondy's theorem, $2^{pq}-1$ has a prime factor p_1 that does not divide 2^p-1 and 2^q-1. Moreover, 2^q-1 has a prime factor p_2 that does not divide 2^p-1. Finally, 2^p-1 has a prime factor p_3.

The next example illustrates a more involved technique of applying Zsigmondy's theorem to solve a class of Diophantine equations.

Example 6 (Balkan MO 2009). Solve the equation $5^x - 3^y = z^2$ in positive integers.

Solution. By considering (mod 3), we see that x must be even. Let $x=2w$. Then $3^y = 5^{2w}-z^2 = (5^w-z)(5^w+z)$. Note that

$$\begin{aligned}(5^w-z,5^w+z) &= (5^w-z,2z)\\ &= (5^w-z,z)\\ &= (5^w,z) = 1,\end{aligned}$$

so $5^w - z = 1$ and $5^w + z = 3^a$ for some positive integer $a \geq 2$. Adding, $2(5^w) = 3^a + 1$. For $a = 2$, we have $w = 1$, corresponding to the solution $x = 2$, $y = 2$ and $z = 4$. For $a \geq 3$, by Zsigmondy's theorem, 3^a+1 has a prime factor p that does not divide $3^2 + 1 = 10$, which implies $p\neq 2$ or 5, so there is no solution in this case.

Example 7. Find all positive integral solutions to $p^a-1=2^n(p-1)$, where p is a prime.

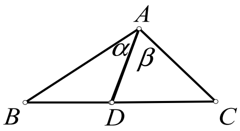
Solution. The case $p=2$ is trivial. Assume p is odd. If a is not a prime, let $a=uv$. Then p^u-1 has a prime factor that does not divide $p-1$. Since p^u-1 divides $p^a-1=2^n(p-1)$, this prime factor of p^u-1 must be 2. But by Zsigmondy's theorem, p^a-1 has a prime factor that does not divide p^u-1 and $p-1$, a contradiction to the equation. So a is a prime. The case $a=2$ yields $p=2^n-1$, i.e. the Mersenne primes. If a is an odd number, then by Zsigmondy's theorem again, $p^a-1=2^n(p-1)$ has a prime factor that does not divide $p-1$; this prime factor must be 2. However, 2 divides $p-1$, a contradiction.

(continued on page 4)

A Geometry Theorem

Kin Y. Li

The following is a not so well known, but useful theorem.



Subtended Angle Theorem. D is a point inside $\angle BAC$ (<180°). Let $\alpha = \angle BAD$ and $\beta = \angle CAD$. D is on side BC if and only if

$$\frac{\sin(\alpha+\beta)}{AD} = \frac{\sin\alpha}{AC} + \frac{\sin\beta}{AB} \qquad (*).$$

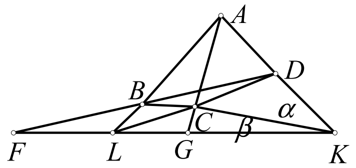
Proof. Note D is on segment BC if and only if the area of $\triangle ABC$ is the sum of the areas of $\triangle ABD$ and $\triangle ACD$. This is

$$\frac{AB\cdot AC\sin(\alpha+\beta)}{2} = \frac{AB\cdot AD\sin\alpha}{2} + \frac{AC\cdot AD\sin\beta}{2}.$$

Multiplying by $2/(AB\cdot AC\cdot AD)$ yields (*).

Below, we will write $PQ\cap RS=X$ to mean lines PQ and RS intersect at point X.

Example 1. Let $AD\cap BC=K$, $AB\cap CD=L$, $BD\cap KL=F$ and $AC\cap KL=G$. Prove that $1/KL = ½(1/KF + 1/KG)$.



Solution. Applying the subtended angle theorem to $\triangle KAL$, $\triangle KDL$, $\triangle KDF$ and $\triangle KAG$, we get

$$\frac{\sin(\alpha+\beta)}{KB} = \frac{\sin\alpha}{KL} + \frac{\sin\beta}{KA}, \quad \frac{\sin(\alpha+\beta)}{KC} = \frac{\sin\alpha}{KL} + \frac{\sin\beta}{KD}$$

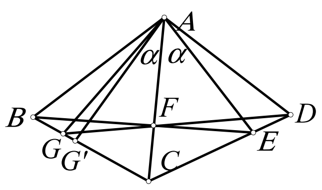
$$\frac{\sin(\alpha+\beta)}{KB} = \frac{\sin\alpha}{KF} + \frac{\sin\beta}{KD}, \quad \frac{\sin(\alpha+\beta)}{KC} = \frac{\sin\alpha}{KG} + \frac{\sin\beta}{KA}$$

Call these (1), (2), (3), (4) respectively. Doing (1)+(2)−(3)−(4), we get

$$0 = \frac{2\sin\alpha}{KL} - \frac{\sin\alpha}{KF} - \frac{\sin\alpha}{KG},$$

which implies the desired equation.

Example 2. (*1999 Chinese National Math Competition*) In the convex quadrilateral $ABCD$, diagonal AC bisects $\angle BAD$. Let E be on side CD such that $BE\cap AC=F$ and $DF\cap BC=G$. Prove that $\angle GAC = \angle EAC$.



Solution. Let $\angle BAC=\angle DAC=\theta$ and G' be on segment BC such that $\angle G'AC = \angle EAC=\alpha$. We will show G', F, D are collinear, which implies $G'=G$. Applying the subtended angle theorem to $\triangle ABE$, $\triangle ABC$ and $\triangle ACD$ respectively, we get

$$(1)\ \frac{\sin(\theta+\alpha)}{AF} = \frac{\sin\alpha}{AB} + \frac{\sin\theta}{AE},$$

$$(2)\ \frac{\sin\theta}{AG'} = \frac{\sin\alpha}{AB} + \frac{\sin(\theta-\alpha)}{AC} \text{ and}$$

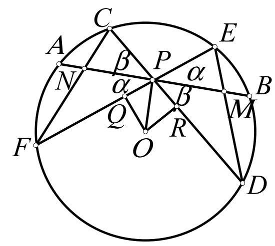
$$(3)\ \frac{\sin\theta}{AE} = \frac{\sin\alpha}{AD} + \frac{\sin(\theta-\alpha)}{AC}.$$

Doing (1)−(2)+(3), we get

$$\frac{\sin(\theta+\alpha)}{AF} = \frac{\sin\alpha}{AD} + \frac{\sin\theta}{AG'}.$$

By the subtended angle theorem, G', F, D are collinear. Therefore, $G = G'$.

Example 3. (*Butterfly Theorem*) Let A,C,E,B,D,F be points in cyclic order on a circle and $CD\cap EF=P$ is the midpoint of AB. Let $M = AB\cap DE$ and $N = AB\cap CF$. Prove that $MP = NP$.



Solution. By the intersecting chord theorem, $PC\cdot PD=PE\cdot PF$, call this x. Applying the subtended angle theorem to $\triangle PDE$ and $\triangle PCF$, we get

$$\frac{\sin(\alpha+\beta)}{PM} = \frac{\sin\alpha}{PD} + \frac{\sin\beta}{PE},$$
$$\frac{\sin(\alpha+\beta)}{PN} = \frac{\sin\alpha}{PC} + \frac{\sin\beta}{PF}.$$

Subtracting these equations, we get

$$\sin(\alpha+\beta)\left(\frac{1}{PM} - \frac{1}{PN}\right) \qquad (*)$$
$$= \sin\beta\,\frac{PF-PE}{x} - \sin\alpha\,\frac{PD-PC}{x}.$$

Let Q and R be the midpoints of EF and CD respectively. Since $OP\perp AB$, we have $PF-PE = 2PQ = 2OP\cos(90°-\alpha) = 2OP\sin\alpha$. Then similarly we have $PD-PC = 2OP\sin\beta$. Hence, the right side of (*) is zero. So the left side of (*) is also zero. Since $0 < \alpha+\beta < 180°$, we get $\sin(\alpha+\beta) \neq 0$. Then $PM = PN$.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***March 28, 2012.***

Problem 386. Observe that $7+1=2^3$ and $7^7+1= 2^3\times113\times911$. Prove that for $n = 2, 3, 4,\ldots$, in the prime factorization of $A_n = 7^{7^n}+1$, the sum of the exponents is at least $2n+3$.

Problem 387. Determine (with proof) all functions $f : [0,+\infty) \to[0,+\infty)$ such that for every $x \ge 0$, we have $4f(x) \ge 3x$ and $f(4f(x) - 3x) = x$.

Problem 388. In $\triangle ABC$, $\angle BAC=30°$ and $\angle ABC=70°$. There is a point M lying inside $\triangle ABC$ such that $\angle MAB=\angle MCA=20°$. Determine $\angle MBA$ (with proof).

Problem 389. There are 80 cities. An airline designed flights so that for each of these cities, there are flights going in both directions between that city and at least 7 other cities. Also, passengers from any city may fly to any other city by a sequence of these flights. Determine the least k such that no matter how the flights are designed subject to the conditions above, passengers from one city can fly to another city by a sequence of at most k flights.

Problem 390. Determine (with proof) all ordered triples (x, y, z) of positive integers satisfying the equation

$$x^2y^2 = z^2(z^2 - x^2 - y^2).$$

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Solutions

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Problem 381. Let k be a positive integer. There are 2^k balls divided into a number of piles. For every two piles A and B with p and q balls respectively, if $p \ge q$, then we may transfer q balls from pile A to pile B. Prove that it is always possible to make finitely many such transfers so as to have all the balls end up in one pile.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **CHAN Chun Wai** and **LEE Chi Man** (Statistics and Actuarial Science Society SS HKUSU), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **Kevin LAU Chun Ting** (St. Paul's Co-educational College, S.3), **LO Shing Fung** (F3E, Carmel Alison Lam Foundation Secondary School) and **Andy LOO** (St. Paul's Co-educational College).

We induct on k. For k=1, we can merge the 2 balls in at most 1 transfer.

Suppose the case $k=n$ is true. For $k=n+1$, since 2^k is even, considering (odd-even) parity of the number of balls in each pile, we see the number of piles with odd numbers of balls is even. Pair up these piles. In each pair, after 1 transfer, both piles will result in even number of balls.

So we need to consider only the situation when all piles have even number of balls. Then in each pile, pair up the balls. This gives altogether 2^n pairs. Applying the case $k=n$ with the paired balls, we solve the case $k=n+1$.

Problem 382. Let $v_0 = 0$, $v_1 = 1$ and

$$v_{n+1} = 8v_n-v_{n-1} \quad \text{for } n = 1,2,3,\ldots.$$

Prove that v_n is divisible by 3 if and only if v_n is divisible by 7.

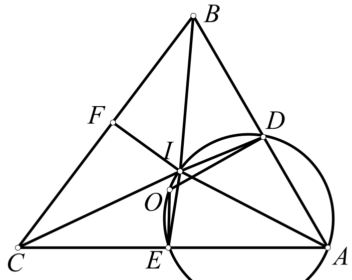
Solution. **Alumni 2011** (Carmel Alison Lam Foundation Secondary School) and **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia) and **Mihai STOENESCU** (Bischwiller, France).

For $n = 1,2,3,\ldots$, $v_{n+2} = 8(8v_n-v_{n-1}) -v_n = 63v_n- 8v_{n-1}$. Then $v_{n+2} \equiv v_{n-1}$ (mod 3) and $v_{n+2} \equiv -v_{n-1}$ (mod 7). Since $v_0= 0$, $v_1=1$, $v_2 = 8$, so v_{3k+1}, $v_{3k+2} \not\equiv 0$ (mod 3) and (mod 7) and $v_{3k} \equiv 0$ (mod 3) and (mod 7).

Other commended solvers: **CHAN Chun Wai** and **LEE Chi Man** (Statistics and Actuarial Science Society SS HKUSU), **CHAN Long Tin** (Diocesan Boys' School), **CHAN Yin Hong** (St. Paul's Co-educational College), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **Kevin LAU Chun Ting** (St. Paul's Co-educational College, S.3), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Andy LOO** (St. Paul's Co-educational College), **NGUYEN van Thien** (Luong The Vinh High School, Dong Nai, Vietnam), **O Kin Chit Alex** (G.T.(Ellen Yeung) College), **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain), **Yan Yin WANG** (City University of Hong Kong, Computing Math, Year 2), **ZOLBAYAR Shagdar** (Orchlon School, Ulaanbaatar, Mongolia),**Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 383. Let O and I be the circumcenter and incenter of $\triangle ABC$ respectively. If $AB\neq AC$, points D, E are midpoints of AB, AC respectively and $BC=(AB+AC)/2$, then prove that the line OI and the bisector of $\angle CAB$ are perpendicular.

***Solution* 1. Kevin LAU Chun Ting** (St. Paul's Co-educational College, S.3).



From $BC = (AB+AC)/2 = BD+CE$, we see there exists a point F be on side BC such that $BF=BD$ and $CF=CE$. Since BI bisects $\angle FBD$, by SAS, $\triangle IBD\cong\triangle IBF$. Then $\angle BDI=\angle BFI$. Similarly, $\angle CEI = \angle CFI$. Then

$$\begin{aligned}&\angle ADI+\angle AEI\\ &= (180°-\angle BDI) + (180°-\angle CEI)\\ &= 360° - \angle BFI - \angle CFI = 180°.\end{aligned}$$

So A,D,I,E are concyclic.

Since $OD \perp AD$ and $OE \perp AE$, so A,D,O,E are also conyclic. Then A,D,I,O are concyclic. So $\angle OIA = \angle ODA=90°$.

***Solution* 2. AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Ercole SUPPA** (Teramo, Italy), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Let $a=BC$, $b=CA$, $c=AB$ and let R, r, s be the circumradius, the inradius and the semiperimeter of $\triangle ABC$ respectively. By the famous formulas $OI^2 = R^2-2Rr$, $s-a = AI\cos(A/2)$, $Rr = abc/(4s)$ and $\cos^2(A/2) = s(s-a)/(bc)$, we get

$$AI^2 = \frac{(s-a)^2}{\cos^2(A/2)} = \frac{bc(s-a)}{s},$$

$$OI^2 = R^2 - 2Rr = R^2 - \frac{abc}{2s}.$$

If $a = (b+c)/2$, then we get $2s = 3a$ and $bc(s-a)/s = abc/(2s)$. So $AI^2+OI^2 = R^2 = OA^2$. By the converse of Pythagoras' Theorem, we get $OI\perp AI$.

Comment: In the last paragraph, all steps may be reversed so that $OI\perp AI$ if and only if $a = (b+c)/2$.

Other commended solvers: **Alumni 2011** (Carmel Alison Lam Foundation Secondary School), **CHAN Chun Wai** and **LEE Chi Man** (Statistics and Actuarial Science Society SS HKUSU), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **Andy LOO** (St. Paul's Co-educational College), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece), **NGUYEN van Thien** (Luong The Vinh High School, Dong Nai, Vietnam), **Mihai STOENESCU** (Bischwiller, France) and **ZOLBAYAR Shagdar** (Orchlon School, Ulaanbaatar, Mongolia).

Problem 384. For all positive real numbers a,b,c satisfying $a + b + c = 3$, prove that

$$\frac{a^2+3b^2}{ab^2(4-ab)}+\frac{b^2+3c^2}{bc^2(4-bc)}+\frac{c^2+3a^2}{ca^2(4-ca)}\geq 4.$$

Solution. **William PENG**.

Let

$$A=\frac{a}{b^2(4-ab)}+\frac{b}{c^2(4-bc)}+\frac{c}{a^2(4-ca)},$$

$$B=\frac{1}{a(4-ab)}+\frac{1}{b(4-bc)}+\frac{1}{c(4-ca)},$$

$$C=\frac{4-ab}{a}+\frac{4-bc}{b}+\frac{4-ca}{c}$$

and $D=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Then A+3B is the left side of the desired inequality. Now since $a + b + c = 3$, we have $C = 4D - 3$. By the Cauchy-Schwarz inequality, we have $(a+b+c)D \geq 3^2$, $AC \geq D^2$ and $BC \geq D^2$. The first of these gives us $D \geq 3$ so that $(D-3)(D-1)\geq 0$, which implies $D^2 \geq 4D-3$. The second and third imply

$$A+3B\geq\frac{4D^2}{C}=\frac{4D^2}{4D-3}\geq 4.$$

Other commended solvers: **Alumni 2011** (Carmel Alison Lam Foundation Secondary School), **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Andy LOO** (St. Paul's Co-educational College), **NGUYEN van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Problem 385. To prepare for the IMO, in everyday of the next 11 weeks, Jack will solve at least one problem. If every week he can solve at most 12 problems, then prove that for some positive integer n, there are n consecutive days in which he can solve a total of 21 problems.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **CHAN Chun Wai** and **LEE Chi Man** (Statistics and Actuarial Science Society SS HKUSU), **Andrew KIRK** (Mearns Castle High School, Glasgow, Scotland), **Andy LOO** (St. Paul's Co-educational College) and **Yan Yin WANG** (City University of Hong Kong, Computing Math, Year 2).

Let S_i be the total number of problems Jack solved from the first day to the end of the i-th day. Since he solves at least one problem everyday, we have $0 < S_1 < S_2 < S_3 < \cdots < S_{77}$. Since he can solve at most 12 problems every week, we have $S_{77} \leq 12\times 11=132$.

Consider the two strictly increasing sequences $S_1, S_2, \cdots, S_{77}$ and $S_1+21, S_2+21, \cdots, S_{77}+21$. Now these 154 integers are at least 1 and at most 132+21=153. By the pigeonhole principle, since the two sequences are strictly increasing, there must be $m < k$ such that $S_k=S_m+21$. Therefore, Jack solved a total of 21 problems from the (m+1)-st day to the end of the k-th day.

Olympiad Corner

(continued from page 1)

Problem 4 (cont). Is it possible to empty both bags after a finite sequence of operations?

Operation b) is now replaced with

b') Triple the number of balls in one bag.

Is it now possible to empty both bags after a finite sequence of operations?

Problem 5. Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers.

Problem 6. Let ABC be an acute-angled triangle. The feet of all the altitudes from A, B and C are D, E and F respectively. Prove that $DE+DF \leq BC$ and determine the triangles for which equality holds.

Zsigmondy's Theorem

(continued from page 2)

<u>Example 8.</u> Find all positive integral solutions to

$$(a+1)(a^2+a+1)\cdots(a^n+a^{n-1}+\cdots+1)$$
$$=a^m+a^{m-1}+\cdots+1.$$

<u>Solution.</u> Note that $n = m = 1$ is a trivial solution. Other than that, we must have $m>n$. Write the equation as

$$\frac{a^2-1}{a-1}\cdot\frac{a^3-1}{a-1}\cdot\ldots\cdot\frac{a^{n+1}-1}{a-1}=\frac{a^{m+1}-1}{a-1},$$

then rearranging we get

$$\left(a^2-1\right)\left(a^3-1\right)\ldots\left(a^{n+1}-1\right)$$
$$=\left(a^{m+1}-1\right)\left(a-1\right)^{n-1}.$$

By Zsigmondy's theorem, we must have $a = 2$ and $m + 1 = 6$, i.e. $m = 5$ (otherwise, a^m-1 has a prime factor that does not divide a^2-1, a^3-1, …, $a^{n+1}-1$, a contradiction), which however does not yield a solution for n.

The above examples show that Zsigmondy's theorem can instantly reduce many number theoretic problems to a handful of small cases. We should bear in mind the exceptions stated in Zsigmondy's theorem in order not to miss out any solutions.

Below are some exercises for the readers.

<u>Exercise 1 (1994 Romanian Team Selection Test).</u> Prove that the sequence $a_n=3^n-2^n$ contains no three terms in geometric progression.

<u>Exercise 2.</u> Fermat's last theorem asserts that for a positive integer $n \geq 3$, the equation $x^n+y^n=z^n$ has no integral solution with $xyz\neq 0$. Prove this statement when z is a prime.

<u>Exercise 3 (1996 British Math Olympiad Round 2).</u> Determine all sets of non-negative integers x, y and z which satisfy the equation $2^x+3^y=z^2$.

<u>Reference</u>

[1] PISOLVE, *The Zsigmondy Theorem,*

http://www.artofproblemsolving.com/Forum/viewtopic.php?f=721&t=422330

[2] Lola Thompson, *Zsigmondy's Theorem*,

http://www.math.dartmouth.edu/~thompson/Zsigmondy's%20Theorem.pdf