The Method of Vieta-Jumping

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Abstract

The *Vieta-Jumping* method, sometimes also called *root flipping*, is a standard procedure for solving easy to recognize types of number theoretic problems. This method has been used extensively in mathematical competitions, most recently at the International Mathematical Olympiad in 2007. The purpose of this article is to analyze this method and to present some of its applications.

The method of Vieta Jumping can be very useful if from the divisibility of some positive integers, some properties of their quotient are asked to be infered. The idea is to assume the existence of a solution for which the statement in question is wrong and then to consider the given relation as a quadratic equation in one of the variables. Using Vieta's formula, we can construct another solution to this equation. The next step is to show that the new solution is valid and smaller than the previous one. By the argument of infinite descent or by assuming the minimality of the first solution, we then obtain a contradiction. To illustrate how this method works, let us solve three classical problems.

The first problem can be considered historical; it was submitted to the IMO in 1988 by the FRG. In [1], Arthur Engel wrote the following note about its difficulty:

> Nobody of the six members of the Australian problem committee could solve it. Two members were Georges Szekeres and his wife, both famous problem solvers and problem creators. Since it was a number theoretic problem it was sent to the four most renowned Australian number theorists. They were asked to work on it for six hours. None of them could solve it in this time. The problem commitee submitted it to the jury of the XXIX IMO marked with a double asterisk, wich meant a superhard problem, possibly too hard to pose. After a long discussion, the jury finally had the courage to choose it as the last problem of the competition. Eleven students gave perfect solutions.

Problem 1 (IMO 1988, Problem 6)**.** *Let a and b be positive integers so that $ab+1$ divides a^2+b^2. Prove that $\frac{a^2+b^2}{ab+1}$ is a perfect square.*

Solution. Let $k = \frac{a^2+b^2}{ab+1}$. Fix k and consider all pairs of nonnegative integers (a, b) which satisfy the equation

$$\frac{a^2+b^2}{ab+1} = k,$$

that is, consider

$$S = \left\{ (a,b) : a, b \in \mathbb{Z}_{\geq 0} \wedge \frac{a^2+b^2}{ab+1} = k \right\}.$$

We claim that among all such pairs in S, there is a pair (a,b) so that $b = 0$ (in this case, we have $k = a^2$ and our proof is complete).

In order to prove this claim, suppose that k is not a perfect square and suppose that $(A,B) \in S$ is the pair which minimizes $a+b$ over all pairs $(a,b) \in S$ (if there exist more than one such pair in S, choose an arbitrary one). Without loss of generality, assume that $A \geq B > 0$. Consider the equation

$$\frac{x^2+B^2}{xB+1} = k,$$

which is equivalent to

$$x^2 - kB \cdot x + B^2 - k = 0$$

as a quadratic equation in x. We know that $x_1 = A$ is one root of this equation. By Vieta's formula, the other root of the equation is

$$x_2 = kB - A = \frac{B^2-k}{A}.$$

The first equation implies that x_2 is an integer, the second that $x_2 \neq 0$ since otherwise, $k = B^2$ would be a perfect square, contradicting our assumption. Also, x_2 cannot be negative, for otherwise,

$$x_2^2 - kBx_2 + B^2 - k \geq x_2^2 + k + B^2 - k > 0,$$

a contradiction. Hence, $x_2 \geq 0$ and thus $(x_2, B) \in S$.

However, because $A \geq B$, we have

$$x_2 = \frac{B^2-k}{A} < A,$$

so $x_2 + B < A + B$, contradicting the minimality of $A + B$. $\square$

Problem 2. *Let x and y be positive integers so that xy divides x^2+y^2+1. Prove that*

$$\frac{x^2+y^2+1}{xy} = 3.$$

Solution. Let $k = \frac{x^2+y^2+1}{xy}$. Fix k and consider all pairs of positive integers (x,y) which satisfy the equation

$$\frac{x^2+y^2+1}{xy} = k.$$

Among all such pairs (x,y), let (X,Y) be a pair which minimizes the sum $x+y$. We claim that $X = Y$. To prove this, assume, for the sake of contradiction, that $X > Y$.

Consider now the equation

$$\frac{t^2 + Y^2 + 1}{tY} = k,$$

which is equivalent to

$$t^2 - kY \cdot t + Y^2 + 1 = 0$$

as a quadratic equation in t. We know that $t_1 = X$ is a root of this equation. The other root can be obtained by Vieta's formula, that is,

$$t_2 = kY - X = \frac{Y^2 + 1}{X},$$

so in particular, t_2 is a positive integer. Also, since $X > Y \geq 1$, we have

$$t_2 = \frac{Y^2 + 1}{X} < X,$$

contradicting the minimality of $X + Y$.

Hence, $X = Y$ and thus, X^2 divides $2X^2 + 1$. Hence, X^2 also divides 1, so $X = 1$ and thus, $k = \frac{X^2+Y^2+1}{XY} = 3$. $\square$

Problem 3 (IMO 2007, Problem 5)**.** *Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.*

Solution. Because $4ab - 1 \mid (4a^2 - 1)^2$, we also have

$$4ab - 1 \mid b^2(4a^2 - 1)^2 - (4ab - 1)(4a^3b - 2ab + a^2) = a^2 - 2ab + b^2 = (a - b)^2.$$

Assume that there exist distinct postive integers a and b so that $4ab - 1$ divides $(a - b)^2$. Let $k = \frac{(a-b)^2}{4ab-1} > 0$. Fix k and let

$$S = \left\{(a, b) : a, b \in \mathbb{Z}^+ \wedge \frac{(a - b)^2}{4ab - 1} = k\right\}$$

and let (A, B) be a pair in S which minimizes the sum $a + b$ over all $(a, b) \in S$. Without loss of generality assume that $A > B$. Consider now the equation

$$\frac{(x - B)^2}{4xB - 1} = k,$$

which is equivalent to

$$x^2 - (2B + 4kB) \cdot x + B^2 + k = 0$$

as a quadratic equation in x, which has roots $x_1 = A$ and, from Vieta's formula,

$$x_2 = 2B + 4kB - A = \frac{B^2 + k}{A}.$$

This implies that x_2 is a positive integer, so $(x_2, B) \in S$. By the minimality of $A + B$, we get $x_2 \geq A$, that is,

$$\frac{B^2 + k}{A} \geq A$$

and therefore $k \geq A^2 - B^2$. Thus

$$\frac{(A - B)^2}{4AB - 1} = k \geq A^2 - B^2$$

and hence,

$$A - B \geq (A + B)(4AB - 1),$$

clearly impossible for $A, B \in \mathbb{Z}^+$. $\square$

References

[1] Arthur Engel, *Problem-Solving Strategies*, Springer, 1999

[2] Mathlinks, *IMO 1988, Problem 6*, `http://www.mathlinks.ro/Forum/viewtopic.php?p=352683`

[3] Mathlinks, $xy|x^2 + y^2 + 1$, `http://www.mathlinks.ro/Forum/viewtopic.php?t=40207`

[4] Mathlinks, *IMO 2007, Problem 5*, `http://www.mathlinks.ro/Forum/viewtopic.php?p=894656`