Secondary Special Camp 2011 : Number Theory Exam

Total time alloted: 5 hours

[Each problem is worth 7 points. The problems are arranged in increasing difficulty.]

Problem 1. (a) Prove that $x^2 + y^2 + z^2 = 2007^{2011}$ has no integer solution. (2 points)

(b) Find all positive integers d such that d divides both $n^2 + 1$ and $(n+1)^2 + 1$ for some integer n. (5 points)

Problem 2. Let n be a positive integer. Prove that the number of *ordered pairs* (a, b) of *relatively prime* positive divisors of n is equal to the number of divisors of n^2.

Problem 3. Prove that $y^2 = x^3 + 7$ has no integer solutions.

Problem 4. Determine all the positive integers $n \geq 3$, such that 2^{2000} is divisible by

$$1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}.$$

Help: (1) $\binom{n}{k} = \dfrac{n!}{k! \cdot (n-k)!}$, where $n! = 1 \cdot 2 \cdot 3 \cdots n$.
(2) Setting $m = n + 1$ might help.
(3) Try to factorize and break the problem into cases.

Date: 19.04.2011