Solutions To Number Theory Problems Of Secondary Special Camp 2011

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Problem 1.

1. Prove that the following equation has no solution in positive integers.
$$x^2 + y^2 + z^2 = 2007^{2011}$$

2. Find all positive integer d such that d divides both n^2+1 and $(n+1)^2+1$ for some natural n.

Solution.

1. We know that, $a^2 \equiv 0, 1$ or $4 \pmod 8$.
First we prove that any number of the form $8k+7$ can't be represented as a sum of three squares.
So, let $x^2 + y^2 + z^2 = 8k + 7$.
$$x^2 \equiv 0, 1, 4 \pmod 8$$
$$y^2 \equiv 0, 1, 4 \pmod 8$$
$$z^2 \equiv 0, 1, 4 \pmod 8$$
This shows that the remainders of $x^2+y^2+z^2$ modulo 8 can be $0, 1, 2, 3, 4, 5, 6$.
But it can never be 7.
Thus, we have proven our claim.
Now, just note that $2007^{2001} \equiv 7 \pmod 8$.
We are done.

2. Here $a|b$ means a divides b. Using this notation,
$d|n^2 + 1$ and $(n+1)^2 + 1$.
So, $d|2n + 1|4n^2 - 1$ and $d|n^2 + 1|4n^2 + 4$.
Thus, we conclude that $d|5$ implying that $d = 1, 5$.

Problem 2. Let n be a positive integer. Prove that the number of **ordered pairs** of (a, b) such that a and b *relatively prime* positive divisors of n is equal to the number of divisors of n^2.

Solution. Denote the number of divisors of n by $\tau(n)$.
Let the canonical prime factorization of n be,
$$n = \prod_{i=1}^{n} p_i^{a_i}$$
Note that every divisor x of n is of the form
$$x = \prod_{i=1}^{n} p_i^{a'_i}$$
where a'_i are non-negative integers. So, $0 \le a'_i \le a_i$. This means that there are $a_i + 1$ choices to have x as a divisor of n. Then, the number of divisors of n,
$$\tau(n) = \prod_{i=1}^{n} (a_i + 1)$$

Now, consider two divisors of n, a and b. They must have some primes of n. Let,

$$a = \prod p_j^{a_j}, b = \prod p_k^{a_k}$$

for some j, k. Note that in the canonical prime factorization of n, we can have a and b co-prime iff a and b both have different prime divisors of n. And obviously, we need to consider a_i positive whereas, a_j and a_k non-negative. Thus for a fixed a_i in the canonical prime factorization of n, we have in total a_i choices to include p_i as a divisor of either a or b, so the number of choice is $2a_i$. Again, since we have to consider non-negative too, we shall have an extra choice for 0. Thus for a fixed a_i, the total choice number is $2a_i + 1$ to have a and b co-prime. Therefore, we may conclude that the number of ways to choose relatively prime ordered pairs of (a, b) is

$$\prod_{i=1}^{n} (a_i + 1).$$

The rest is to just note that,

$$\tau(n^2) = \prod_{i=1}^{n} (2a_i + 1).$$

Problem 3. Prove that $y^2 = x^3 + 7$ has no integer solutions.

Solution. If x even, $y^2 \equiv -1 \pmod 4$, so contradiction! Therefore, x odd. Then $y^2 + 1 = (x + 2)(x^2 - 2x + 4)$

Lemma 1. *Every odd prime divisor of $n^2 + 1$ is of the form $4k + 1$.*

It is a very useful one in number theory.
Proof :
$n^2 \equiv -1 \pmod p \Longrightarrow n^4 \equiv 1 \pmod p$.
Also, by Fermat's little theorem, $n^{p-1} \equiv 1 \pmod p$.
Thus, $4|p - 1 \Longrightarrow p \equiv 1 \pmod 4$.

Corollary :
If $p|a^2 + b^2$ with $p \equiv 3 \pmod 4$, then $p|a, p|b$.

Now, come back to the original problem.
Consider two cases.

#1. $x \equiv 1 \pmod 4$.
Then $x + 2 \equiv 3 \pmod 4$ but it will have an odd prime of the form $4k - 1$ an odd times, contradiction!

#2. If $x \equiv 1 \pmod 4), x^2 - 2x + 4 \equiv -1 \pmod 4$, again contradiction.
But $4k - 1 \not| 1$, according to the corollary a contradiction follows and thus the equation has no solution in $\mathbb{N}$.

Problem 4. Determine all the positive integers $n \geq 3$, such that 2^{2000} is divisible by

$$1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$$

Note. This is a problem from 1998 Chinese National Olympiad.

Solution. Obviously $n \geq 3$.
Since 2 is a prime, we need

$$1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} = 2^k$$

for some $k \in \mathbb{N}, k \leq 2000$. Note that,

$$1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} = \frac{(n+1)(n^2 - n + 6)}{6}$$

So,

$$(n+1)(n^2 - n + 6) = 3 \cdot 2^{k+1}$$

We have two cases:

#1. $n + 1$ is a power of 2 i.e. $n + 1 = 2^r$.
Because $n + 1 \geq 4, r \geq 2$. We get,
$2^{2r} - 2^{r+1} + 1 - 2^r + 1 + 6 = 3 \cdot 2^s$ where $s = k + 1 - r$.
$\Longrightarrow 2^{2r} - 3 \cdot 2^r + 8 = 3 \cdot 2^s$
If $r \geq 4$, $3 \cdot 2^s \equiv 8 \pmod{16} \Longrightarrow s = 3$.
But then $2^r(2^r - 3) = 16 \Longrightarrow 2^r - 3 = 1, r = 2$ which is not a solution.
So, $r = 2, 3$ which gives the solution $n = 7, 3$.

#2. $n + 1 = 3 \cdot 2^a$ for some $a \in \mathbb{N}$
Now, $a \geq 1$ and $9 \cdot 2^{2a} - 9 \cdot 2^a + 8 = 2^t$ for some t.
Then if $a > 3$, $9 \cdot 2^{2a} - 9 \cdot 2^a + 8 = 2^3(9 \cdot 2^{2a-3} - 9 \cdot 2^{a-3} + 1)$ would have an odd factor namely $9 \cdot 2^{2a-3} - 9 \cdot 2^{a-3} + 1$.
But obviously $a \neq 1, 2$ which gives $a = 3, n = 23$.

Thus $n = 3, 7, 23$ are all solutions.