Problem (APMO - Problem 3)**.** Find all pairs of (n, p) so that $\dfrac{n^p+1}{p^n+1}$ is a positive integer where n is a positive integer and p is a prime number.

Solution. We can re-state the relation as

$$p^n+1|n^p+1$$

Firstly, we exclude the case $p=2$. In this case,

$$2^n+1|n^2+1$$

Obviously, we need

$$n^2+1 \geq 2^n+1 \Rightarrow n^2 \geq 2^n$$

But, using induction we can easily say that for $n>4$, $2^n>n^2$ giving a contradiction. Checking $n=1,2,3,4$ we easily get the solutions:

$$(n,p)=(2,2),(4,2)$$

We are left with p odd. So, p^n+1 is even, and hence n^p+1 as well. This forces n to be odd. Say, q is an arbitrary prime factor of $p+1$. If $q=2$, then $q|n+1$ and since

$$n^p+1=(n+1)(n^{p-1}-....+1)$$

and p odd, there are p terms in the right factor, therefore odd. So, we infer that $2^k|n+1$ where k is the maximum power of 2 in $p+1$.

We will use the following lemmas without proof for being well-known.

LEMMA 1. *If $a|b$ and $a|c$, then $a|\gcd(b,c)$.*

LEMMA 2. *If*

$$a^x \equiv b^x \pmod{n}$$

and,

$$a^y \equiv b^y \pmod{n}$$

then

$$a^{\gcd(x,y)} \equiv b^{\gcd(x,y)} \pmod{n}$$

LEMMA 3.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

where e is the Euler constant.

Now, we prove the following lemmas.

LEMMA 4. *If x is the smallest positive integer such that*

$$a^x \equiv 1 \pmod{n}$$

then if,

$$a^m \equiv 1 \pmod{n}$$

m is divisible by x.

Proof. Let, $m = xk + r$ with $r < x$. Then, since $a^x \equiv 1$,

$$a^m \equiv (a^x)^k \cdot a^r \equiv 1$$

This implies,

$$a^r \equiv 1 \pmod{n}$$

But this is a contradiction for the minimum $x > r$. So, we must have $r = 0$ that is, $x|m$.

□

LEMMA 5. *If $g = \gcd\left(a+1, \frac{a^p+1}{a+1}\right)$, then $g|p$.*

PROOF:

$$\frac{a^p+1}{a+1} = (a^{p-1} - a^{p-2} \ldots - a + 1)$$

From Euclid's algorithm,

$$\gcd\left(a+1, \frac{a^p+1}{a+1}\right) = \gcd(a+1, (-1)^{p-1} - (-1)^{p-2} + .. + 1) = \gcd(a+1, p)$$

□

LEMMA 6. *If p is an odd prime, then $p^n \leq n^p$ for $p \leq n$.*

PROOF. This is true for $n = 1$. Say, this is also true for some smaller values of n. Now, we prove this for $n+1$.

Since $p \leq n$,

$$(pn + p)^p \leq (pn + n)^p$$

and therefore,

$$(n+1)^p = n^p(1 + \frac{1}{n})^p \leq p^n(1 + \frac{1}{p})^p \leq p^n \cdot e < p^{n+1}$$

□

Back to the problem. Assume that q is odd.

$$q|p^n+1|n^p+1$$

Write them using congruence. And we have,

$$n^p \equiv -1 \pmod q$$

$$\Rightarrow n^{2p} \equiv 1 \pmod q$$

Suppose, $e = ord_q(n)$ i.e. e is the smallest positive integer such that

$$n^e \equiv 1 \pmod q$$

Then, $e|2p$ and $e|q-1$ from lemma 4.

Also, from Fermat's theorem,

$$n^{q-1} \equiv 1 \pmod q$$

Therefore,

$$n^{\gcd(2p,q-1)} \equiv 1 \pmod q$$

From p odd and $q|p+1$, $p > q$ and so p and $q-1$ are co-prime. Thus,

$$\gcd(2p,q-1) = \gcd(2,q-1) = 2$$

From lemma 1, $e|\gcd(2p,q-1)$ and so we must have $e=2$. Again, since p odd, if $p = 2r+1$,

$$n^{2r+1} \equiv n \pmod q$$

Hence, $q|n+1$. If $q|\frac{n^p+1}{n+1}$, then by the lemma above we get

$$q|\gcd\left(n+1, \frac{n^p+1}{n+1}\right)|p$$

which would imply $q=1$ or p. Both of the cases are impossible. So, if s is the maximum power of q so that $q^s|p+1$, then we have $q^s|n+1$ too for every prime factor q of $p+1$. This leads us to the conclusion $p+1|n+1$ or $p \le n$ which gives $p^n \ge n^p$. But from the given relation,

$$p^n+1 \le n^p+1 \Rightarrow p^n \le n^p$$

Combining these two, $p=n$ is the only possibility to happen.

Thus, the solutions are

$$(n,p) = (2,4),(p,p)$$