

Number Theory Problems From IMO 1959-2012

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ABSTRACT. It's been on my planning list for a long time, to make this document. Finally I was able to compile all the number theory problems that appeared at the IMO so far. However, this time I am only compiling the problems. On a later version, I will combine their solutions as well. Thanks to the great **IMO Compendium**. Each problem has the year and chronological number of its appearance with the problem number.

1. PROBLEMS

1 (1959, 1). Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every $n \in \mathbb{N}$.

Proposed by-Poland

2 (1960, 1). Find all the three-digit numbers for which one obtains, when dividing the number by 11, the sum of the squares of the digits of the initial number.

Proposed by-Bulgaria

3 (1962, 1). Find the smallest natural number n with the following properties:

- (1) In decimal representation it ends with 6.
- (2) If we move this digit to the front of the number, we get a number 4 times larger.

Proposed by-Poland

4 (1964, 1). (a). Find all natural numbers n such that 7 divides $2^n - 1$.

(b). Prove that for all natural numbers n the number $2^n + 1$ is not divisible by 7.

Proposed by-Czechoslovakia

5 (1967, 3). Let k, m , and n be positive integers such that $m+k+1$ is a prime number greater than $n+1$. Write $c_s = s(s+1)$. Prove that the product

$$(c_{m+1} - c_k)(c_{m+2} - c_k) \cdots (c_{m+n} - c_k)$$

is divisible by $c_1 c_2 \cdots c_n$.

Proposed by-United Kingdom

6 (1968, 2). Find all positive integers x for which

$$p(x) = x^2 - 10x - 22$$

where $p(x)$ denotes the product of the digits of x .

Proposed by-Czechoslovakia

7 (1968, 6). Let $[x]$ denote the integer part of x , i.e., the greatest integer not exceeding x for positive x . If n is a positive integer, express as a simple function of n the sum

$$\left[\frac{n+1}{2} \right] + \left[\frac{n+2}{4} \right] + \dots + \left[\frac{n+2^i}{2^{i+1}} \right]$$

Proposed by-United Kingdom

8 (1969, 1). Prove that there exist infinitely many natural numbers a with the following property: the number

$$z = n^4 + a$$

is composite for any natural n .

Proposed by-Germany, DR

9 (1970, 2). Let a and b be the bases of two number systems and let

$$\begin{aligned} A_n &= \overline{x_1 x_2 \dots x_n}_{(a)}, & A_{n+1} &= \overline{x_0 x_1 x_2 \dots x_n}_{(a)} \\ B_n &= \overline{x_1 x_2 \dots x_n}_{(b)}, & B_{n+1} &= \overline{x_0 x_1 x_2 \dots x_n}_{(b)} \end{aligned}$$

be numbers in the number systems with respective bases a and b , so that $x_0, x_1, x_2, \dots, x_n$ denote digits in the number system with base a as well as in the number system with base b . Suppose that neither x_0 nor x_1 is zero. Prove that $a > b$ if and only if

$$\frac{A_n}{A_{n+1}} > \frac{B_n}{B_{n+1}}$$

Proposed by-Romania

10 (1970, 4). For what natural numbers n can the product of some of the numbers $n, n+1, n+2, n+3, n+4, n+5$ be equal to the product of the remaining ones?

Proposed by-Czechoslovakia

11 (1971, 3). Prove that the sequence $2^n - 3, (n > 1)$ contains a sub-sequence of numbers relatively prime in pairs.

Proposed by-Poland

12 (IMO 1972, 1). A set of 10 positive integers is given such that the decimal expansion of each of them has two digits. Prove that there are two disjoint subsets of the set with equal sums of their elements.

Proposed by-Soviet Union

13 (1972, 3). Let m and n be nonnegative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer.

Proposed by-United Kingdom

14 (1974, 3). Prove that there does not exist a natural number n for which the number

$$5 \nmid \sum_{k=0}^n \binom{2n+1}{2k+1} 2^{3k}$$

Proposed by-Romania

15 (1974, 6). Let $P(x)$ be a polynomial with integer coefficients. If $n(P)$ is the number of (distinct) integers k such that $P^2(k) = 1$, prove that $n(P) - \deg(P) \geq 2$, where $\deg(P)$ denotes the degree of the polynomial P .

Proposed by-Sweden

16 (1975, 2). Let a_1, a_2, a_3, \dots be any infinite increasing sequence of positive integers. (For every integer $i > 0, a_{i+1} > a_i$.) Prove that there are infinitely many m for which positive integers x, y, h, k can be found such that $0 < h < k < m$ and

$$a_m = xa_h + ya_k$$

Proposed by-United Kingdom

17 (1975, 4). Let A be the sum of the digits of the number 4444^{4444} and B the sum of the digits of the number A . Find the sum of the digits of the number B .

Proposed by-Soviet Union

18 (1976, 2). Let $P_1(x) = x^2 - 2$ and

$$P_j(x) = P_1(P_{j-1}(x)), j = 2, 3, \dots$$

Show that for arbitrary n , the roots of the equation

$$P_n(x) = x$$

are real and different from one another.

Proposed by-Finland

19 (1976, 4). Find the largest number obtainable as the product of positive integers whose sum is 1976.

Proposed by-USA

20 (1977, 2). In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

proposed by-Vietnam

21 (1977, 3). Let n be a given integer greater than 2, and let V_n be the set of integers $1 + kn$, where $k = 1, 2, \dots$. A number $m \in V_n$ is called indecomposable in V_n if there do not exist numbers $p, q \in V_n$ such that $pq = m$. Prove that there exists a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Expressions that differ only in order of the elements of V_n will be considered the same.)

Proposed by-Netherlands

22 (1977, 5). Let a and b be natural numbers and let q and r be the quotient and remainder respectively when $a^2 + b^2$ is divided by $a + b$. Determine the numbers a and b if

$$q^2 + r = 1977$$

Proposed by-Germany, FR

23 (1977, 6). Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function that satisfies the inequality

$$f(n+1) > f(f(n))$$

for all $n \in \mathbb{N}$. Prove that $f(n) = n$ for all natural numbers n .

Proposed by-Bulgaria

24 (1978, 1). Let $n > m \geq 1$ be natural numbers such that the groups of the last three digits in the decimal representation of $1978m, 1978n$ coincide. Find the ordered pair (m, n) of such m, n for which $m + n$ is minimal.

Proposed by-Cuba

25 (1978, 3). Let $f\{(n)\}$ be a strictly increasing sequence of positive integers: $0 < f(1) < f(2) < f(3) < \dots$. Of the positive integers not belonging to the sequence, the n th in order of magnitude is $f(f(n)) + 1$. Determine $f(240)$.

Proposed by-United Kingdom

26 (1978, 5). Let $\varphi : \{1, 2, 3, \dots\} \rightarrow \{1, 2, 3, \dots\}$ be injective. Prove that for all n ,

$$\sum_{k=1}^n \frac{\varphi(k)}{k^2} \geq \sum_{k=1}^n \frac{1}{k}$$

Proposed by-France

27 (1979, 1). Given that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{1318} + \frac{1}{1319} = \frac{p}{q}$$

for co-prime p, q . Prove that, $1979|p$.

Proposed by-Germany, FR

28 (1981, 3). Determine the maximum value of $m^2 + n^2$ where m and n are integers satisfying

$$m, n \in \{1, 2, \dots, 1981\} \quad \text{and} \quad (n^2 - mn - m^2)^2 = 1$$

Proposed by-Netherlands

29 (1981, 6). Assume that $f(x, y)$ defined for all positive integers x and y , and that the following equations are satisfied:

$$\begin{aligned} f(0, y) &= y + 1, \\ f(x + 1, 0) &= f(x, 1), \\ f(x + 1, y + 1) &= f(x, f(x + 1, y)) \end{aligned}$$

Determine $f(4, 1981)$.

Proposed by-Finland

30 (1982, 1). The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n ,

$$f(m + n) - f(m) - f(n) = 0 \text{ or } 1$$

$$f(2) = 0, f(3) > 0 \text{ and } f(9999) = 3333$$

Determine $f(1982)$.

Proposed by-United Kingdom

31 (1982, 4). Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x, y) , then it has at least three such solutions. Show that the equation has no solution in integers when $n = 2891$.

Proposed by-United Kingdom

32 (1983, 3). Let a, b, c be positive integers s.t. $(a, b) = (b, c) = (c, a) = 1^1$. Show that $2abc - ab - bc - ca$ is the largest integer not representable as

$$bcx + cay + abz$$

for non-negative integer x, y, z .

Proposed by-Germany, FR

33 (1983, 5). Prove or disprove the following statement: In the set $\{1, 2, 3, \dots, 10^5\}$ a subset of 1983 elements can be found that does not contain any three consecutive terms of an arithmetic progression.

Proposed by-Poland

34 (1984, 2). Find two positive integers a, b such that none of the numbers $a, b, a + b$ is divisible by 7 but $(a + b)^7 - a^7 - b^7$ is.

Proposed by-Netherlands

35 (1984, 6). Let a, b, c, d be odd positive integers such that $a < b < c < d, ad = bc$, and

$$a + d = 2^k, b + c = 2^m$$

for some positive integer k, m . Prove that, $a = 1$.

Proposed by-Poland

36 (1985, 3). The weight $w(p)$ of a polynomial p ,

$$p(x) = \sum_{i=0}^n a_i x^i$$

with integer coefficients a_i defined as the number of its odd coefficients. For $i = 0, 1, 2, \dots$ let

$$q_i(x) = (1 + x)^i$$

Prove that for any finite sequence $0 \leq i_1, i_2, \dots, i_n$,

$$w(q_{i_1} + q_{i_2} + \dots + q_{i_n}) \geq w(q_{i_1})$$

Proposed by-Netherlands

37 (1985, 4). Given a set M of 1985 positive integers, none of which has a prime divisor larger than 26, prove that M has four distinct elements whose geometric mean is an integer.

Proposed by-Mongolia

¹ (a, b) is the greatest common divisor of a and b

38 (1986, 1). The set $S = \{2, 5, 13\}$ has the property that for every $a, b \in S, a \neq b$, the number $ab-1$ is a perfect square. Show that for every positive integer d not in S , the set $S \cup \{d\}$ does not have the above property.

Proposed by-Germany, FR

39 (1987, 1). Let S be a set of n elements. We denote the number of all permutations of S that have exactly k fixed points by $p_n(k)$. Prove that

$$\sum_k p_n(k) = n!$$

Proposed by-Germany, FR

40 (1987, 4). Does there exist a function $f : \mathbb{N} \rightarrow \mathbb{N}$, such that

$$f(f(n)) = n + 1987$$

for every natural number n ?

Proposed by-Soviet Union

41 (1987, 6). Let

$$f(x) = x^2 + x + p, p \in \mathbb{N}$$

Prove that if the numbers $f(0), f(1), \dots, f\left(\left[\frac{p}{3}\right]\right)$ are primes, then all the numbers $f(0), f(1), \dots, f(p-2)$ are primes.

Proposed by-Soviet Union

42 (1988, 3). A function f defined on the positive integers (and taking positive integer values) is given by

$$\begin{aligned} f(1) &= 1, & f(3) &= 3 \\ f(2n) &= f(n) \\ f(4n+1) &= 2f(2n+1) - f(n) \\ f(4n+3) &= 3f(2n+1) - 2f(n) \end{aligned}$$

for all positive integers n . Determine with proof the number of positive integers n less than or equal to 1988 for which $f(n) = n$.

Proposed by-United Kingdom

43 (1988, 6). ² Let a and b be two positive integers so that

$$k = \frac{a^2 + b^2}{ab + 1} \in \mathbb{N}$$

Prove that k is a square.

²This is one of the most historical problems in IMO.

Proposed by-Germany, FR

44 (1989, 5). For which positive integers n does there exist a positive integer N such that none of the integers $1 + N, 2 + N, \dots, n + N$ is the power of a prime number?

Proposed by-Sweden

45 (1990, 3). Find all positive integers n s.t.

$$n^2 | 2^n + 1$$

Proposed by-Romania

46 (1990, 5). Two players A and B play a game in which they choose numbers alternately according to the following rule: At the beginning, an initial natural number $n_0 > 1$ is given. Knowing n_{2k} , player A may choose any $n_{2k+1} \in \mathbb{N}$, such that

$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2$$

Then player B chooses a number $n_{2k+2} \in \mathbb{N}$ such that

$$\frac{n_{2k+1}}{n_{2k+2}} = p^r$$

where $p \in \mathbb{P}, r \in \mathbb{N}$.³

It is stipulated that player A wins the game if he (she) succeeds in choosing the number 1990, and player B wins if he (she) succeeds in choosing 1. For which natural numbers n_0 can player A manage to win the game, for which n_0 can player B manage to win, and for which n_0 can players A and B each force a tie?

Proposed by-Germany, FR

47 (1991, 2). Let $n > 6$ and let $a_1 < a_2 < \dots < a_k$ be all natural numbers that are less than n and relatively prime to n . Show that if a_1, a_2, \dots, a_k is an arithmetic progression, then n is a prime number or a natural power of two.

Proposed by-Romania

48 (1992, 1). Find all integer triples (p, q, r) such that $1 < p < q < r$ and $(p-1)(q-1)(r-1)$ is a divisor of $(pqr-1)$.

Proposed by-New Zealand

49 (1992, 6). For each positive integer n , denote by $s(n)$ the greatest integer such that for all positive integers $k \leq s(n)$, n^2 can be expressed as a sum of squares of k positive integers.

- (i) Prove that $s(n) \leq n^2 - 14$ for all n .
- (ii) Find a number n s.t. $s(n) = n^2 - 14$.

³ \mathbb{P} is the set of primes.

(iii) Prove that there are infinitely many n s.t. $s(n) = n^2 - 14$.

Proposed by-United Kingdom

50 (1993, 1). Let $n > 1$ be an integer and $f(x) = x^n + 5x^{n-1} + 3$. Prove that there do not exist polynomials $g(x), h(x)$, each having integer coefficients and degree at least one, such that

$$f(x) = g(x)h(x)$$

Proposed by-Ireland

51 (1993, 5). Determine whether there exists a strictly increasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ with the following properties:

$$\begin{aligned} f(1) &= 2 \\ f(f(n)) &= f(n) + n \end{aligned}$$

Proposed by-Germany

52 (1994, 1). Let m and n be positive integers. The set $A = \{a_1, a_2, \dots, a_m\}$ is a subset of $1, 2, \dots, n$. Whenever $a_i + a_j \leq n, 1 \leq i \leq j \leq m, a_i + a_j$ also belongs to A . Prove that

$$\frac{a_1 + a_2 + \dots + a_m}{m} \geq \frac{n+1}{2}$$

Proposed by-France

53 (1994, 3). For any positive integer k, A_k is the subset of

$$\{k+1, k+2, \dots, 2k\}$$

consisting of elements whose digit in base 2 contain exactly three 1's. Let $f(k)$ denote the number of elements of A_k .

(a) Prove that for any positive integer $m, f(k) = m$ has at least one solution.

(b) Determine all positive integers m for which $f(k) = m$ has a unique solution.

Proposed by-Romania

54 (1994, 4). Determine all pairs (m, n) of positive integers such that

$$\frac{n^3 + 1}{mn + 1} \in \mathbb{N}$$

Proposed by-Australia

55 (1994, 6). Find a set A of positive integers such that for any infinite set P of prime numbers, there exist positive integers $m \in A$ and $n \notin A$, both the product of the same number (at least two) of distinct elements of P .

Proposed by-Finland

56 (1995, 6). Let p be an odd prime. Find the number of p -element subsets A of $1, 2, \dots, 2p$ such that the sum of all elements of A is divisible by p .

Proposed by-Poland

57 (1996, 3). Let \mathbb{N}_0 denote the set of non-negative integers. Find all $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ s.t.

$$f(m + f(n)) = f(f(m)) + f(n) \quad \text{for all } n, m \in \mathbb{N}_0$$

Proposed by-Romania

58 (1996, 4). The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

Proposed by-Russia

59 (1996, 6). Let p, q, n be three positive integers s.t. $p + q < n$. Let (x_0, x_1, \dots, x_n) be a $n + 1$ -tuple of integers satisfying

(a) $x_0 = x_n = 0$

(b) For each i with $1 \leq i \leq n$, $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$. Show that there exists a pair (i, j) of distinct indices with $(i, j) \neq (0, n)$ such that $x_i = x_j$.

Proposed by-France

60 (1997, 5). Find all pairs of integers x, y s.t.

$$x^{y^2} = y^x$$

Proposed by-Czechoslovakia

61 (1997, 6). For a positive integer n , let $f(n)$ denote the number of ways to represent n as a sum of non-negative integer powers of 2. Representations that differ only in the ordering in their summation are not considered to be distinct (For example, $f(4) = 4$). Prove the inequality

$$2^{\frac{n^2}{4}} \leq f(n) \leq 2^{\frac{n^2}{2}}$$

Proposed by-Lithuania

62 (1998, 3). For any $n \in \mathbb{N}$, $\tau(n)$ denotes the number of positive divisors of n . Determine all positive integers m s.t.

$$m = \frac{\tau(k^2)}{\tau(k)}$$

has a solution.

Proposed by-Belarus

63 (1998, 4). Determine all pairs (x, y) of positive integers such that

$$xy^2 + y + 7 \mid x^2y + x + y$$

Proposed by-United Kingdom

64 (1998, 6). Determine the least possible value of $f(1998)$, where f is a function from the set \mathbb{N} of positive integers into itself such that for all $m, n \in \mathbb{N}$,

$$f(n^2 f(m)) = m (f(n)^2)$$

Proposed by-Bulgaria

65 (1999, 4). Find all pairs of positive integers (x, p) such that $x \leq 2p, p \in \mathbb{P}$, and

$$x^{p-1} \mid (p-1)^x + 1$$

Proposed by-Taiwan

66 (2000, 5). Does there exist a positive integer n such that n has exactly 2000 prime divisors and $n \mid 2^n + 1$.

Proposed by-Russia

67 (2001, 4). Let n be an odd positive integer greater than 1 and c_1, c_2, \dots, c_n be integers. For each permutation $a = (a_1, a_2, \dots, a_n)$ of $(1, 2, \dots, n)$ define

$$S(a) = \sum_{i=1}^n c_i a_i$$

Prove that, there exists permutations a, b s.t. $n! \mid S(a) - S(b)$.

Proposed by-Canada

68 (2001, 6). Let $a > b > c > d$ be positive integers and suppose

$$ac + bd = (b + d + a - c)(b + d - a + c)$$

Prove that, $ad + bc$ isn't a prime.

Proposed by-Bulgaria

69 (2002, 3). Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$a^n + a^2 - 1 \mid a^m + a - 1$$

Proposed by-Romania

70 (2002, 4). Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \dots < d_k = n$. Prove that

$$d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$$

is always less than n^2 , and determine when it is a divisor of n^2 .

Proposed by-Romania

71 (2003, 2). Determine all pairs (a, b) of positive integers such that

$$\frac{a^2}{2ab^2 - b^3 + 1} \in \mathbb{N}$$

Proposed by-Bulgaria

72 (2003, 6). Let p be a prime number. Prove that there exists a prime number q such that for every integer n , the number $n^p - p$ is not divisible by q .

Proposed by-France

73 (2004, 6). We call a positive integer *alternate* if its decimal digits are alternately odd and even. Find all positive integers n such that n has an alternate multiple.

Proposed by-Iran

74 (2005, 2). Let a_1, a_2, \dots be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer n , the numbers a_1, a_2, \dots, a_n gives different remainder upon division by n . Prove that each integer occurs exactly once in the sequence.

Proposed by-Holland

75 (2006, 4). Consider the sequence a_1, a_2, \dots defined by

$$a_n = 2^n + 3^n + 6^n - 1$$

Determine all positive integers that are relatively prime to every term of the sequence.

Proposed by-Poland

76 (2006, 4). Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2$$

Proposed by-

77 (2006, 5). Let $P(x)$ be a polynomial in x with $\deg(P) = n > 1$ and $k \in \mathbb{N}$.

$$Q(x) = P(P(\dots(P(x))\dots))$$

where P occurs k times. Prove that there are at most n integers t such that

$$Q(t) = t$$

Proposed by-

78 (2007, 5). Let a and b be positive integers s.t.

$$4ab - 1 \mid (4a^2 - 1)^2$$

Prove that, $a = b$.

Proposed by-

79 (2008, 3). Prove that there are infinite n s.t. $n^2 + 1$ has a prime factor greater than $2n + \sqrt{2n}$.

Proposed by-

80 (2009, 1). Let n be a positive integer and let a_1, \dots, a_k ($k \geq 2$) be distinct integers in the set $\{1, 2, \dots, n\}$ s.t.

$$n \mid a_i(a_{i+1} - 1)$$

for $i = 1, 2, \dots, k - 1$. Prove that $n \mid a_k(a_1 - 1)$.

Proposed by-

81 (2009, 3). Let s_1, s_2, \dots be a strictly increasing sequence of positive integers. If

$$s_{s_1}, s_{s_2}, \dots, s_{s_n} \quad \text{and} \quad s_{s_1+1}, s_{s_2+1}, \dots, s_{s_n+1}$$

are both arithmetic sequences, then show that, s_1, s_2, \dots it-self is an arithmetic sequence.

Proposed by-

82 (2009, 5). Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b , there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b + f(a) - 1)$$

(A triangle is non-degenerate if its vertices are not collinear.)

83 (2010, 1). *Proposed by-*

84 (2010, 3). Find all

Proposed by-

85 (2011, 1). Given any set

$$A = \{a_1, a_2, a_3, a_4\}$$

of four distinct positive integers, we denote the sum $s_A = a_1 + a_2 + a_3 + a_4$. Let n_A be the number of pairs (i, j) with $1 \leq i \leq j \leq 4$ s.t. $a_i + a_j \mid s_A$. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A .

Proposed by-

86 (2011, 5). Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n ,

$$f(m - n) \mid f(m) - f(n)$$

Prove that, for all m, n with $f(m) \leq f(n)$, $f(m) \mid f(n)$.

Proposed by-

87.