Practice Problem: ONTC Day 2

August 26, 2015

There are all sorts of problems. You don't have to send solutions for practice problems. Just comment on the post.

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1. Practice Problems

Problems are not in increasing order of difficulties.

Problem 1.1. Determine the largest n so that $n + 5 | n^4 + 1395$.

Problem 1.2. Determine the number of positive integers smaller than 1000000, that are also perfect squares and give a remainder 4 when divided by 8.

Problem 1.3. Does there exist a positive integer n such that $n^2 + 2n + 2015$ is a perfect square?

Problem 1.4. Given are positive integers r and k and an infinite sequence of positive integers $a_1 \leq a_2 \leq \ldots$ such that $\frac{i}{a_i} = k + 1$. Prove that there is a j so that $\frac{j}{a_j} = k$.

Problem 1.5. A positive integer is called *wacky* if its decimal representation contains 100 digits, and if by removing any of those digits one gets a 99-digit number divisible by 7. How many wacky positive integers are there?

Problem 1.6. We say that a positive integer is an *almost square* if it is a product of two positive integers differing by 1. Prove that every almost square is a ratio of two almost squares.

Problem 1.7. Does there exist an infinite sequence of positive integers such that for every positive integer k, the sum of every k consecutive terms of this sequence is divisible by $k+1$?

Problem 1.8. We say that a positive integer is *interesting* if the sum of its digits is a prime number. Determine the greatest possible number of interesting numbers which may appear among five consecutive positive integers.

Problem 1.9. Integers $a, x_1, x_2, ..., x_{13}$ satisfy the relation:

$$a = (1 + x_1)(1 + x_2) \cdots (1 + x_{13}) = (1 - x_1)(1 - x_2) \cdots (1 - x_{13})$$

Prove that, $ax_1x_2 \cdots x_{13} = 0$.

Problem 1.10. Let $n > 1$ be an integer. Consider all the fractions $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$ and reduce each of them to an irreducible form. Denote the sum of numerators of the obtained fractions by $f(n)$. Find all n such that $f(n) + f(2015n)$ is odd.

Problem 1.11. Let a and b be two positive integers satisfying $gcd(a, b) = 1$. Consider a pawn standing on the grid point (x, y). A step of type A consists of moving the pawn to one of the following grid points: $(x+a, y+a), (x+a, y-a), (x-a, y+a), (x-a, y-a)$. A step of type B consists of moving the pawn to $(x+b, y+b), (x+b, y-b), (x-b, y+b), (x-b, y-b)$. Now put a pawn on $(0, 0)$. You can make a (finite) number of steps, alternatingly of type A and type B, starting with a step of type A. You can make an even or odd number of steps, i.e., the last step could be of either type A or type B. Determine the set of all grid points (x, y) that you can reach with such a series of steps.